A hidden symmetry of collisionless sound of Bose condensates in anisotropic traps

Martin Fliesser and Robert Graham
Fachbereich Physik, Universität-Gesamthochschule Essen
45117 Essen, Germany

Abstract

We derive a symmetry property for the Fourier-transform of the collisionless sound modes of Bose condensates in anisotropic traps connected with a somewhat hidden conservation law. We discuss its possible observation by dispersive light scattering.

PACS: 03.75.Fi,05.30.Jp,42.50.Gy

Since the achievement of Bose-Einstein condensates in trapped alkali gases [1–3] at ultralow temperatures numerous aspects of these systems have been investigated both from the experimental and the theoretical side. Adding a time dependent perturbation to the trap potential the lowest frequencies of the density oscillation spectrum were measured by observing condensate shape oscillations either by time of flight measurements [4–6] or by dispersive light scattering [7]. In this paper we want to investigate these mode functions in detail and show how they reflect a special symmetry of the system. We disuss how this symmetry property could be observed in phase-contrast imaging measurements [7,8] and comment on the feasibility of its observation in inelastic light sacttering.

We restrict ourselves to the density fluctuations of energies much smaller than the chemical potential ($\hbar\omega \ll \mu$) of a trapped Bose condensate at temperature $T = 0$. The equations governing these collisionless sound modes in a weakly interacting system in mean field approximation can be written in the form of hydrodynamic equations [9,10]. For isotropic [9], axially symmetric [14] and even fully anisotropic harmonic traps [16] the solutions of these equations can be classified in terms of three quantum numbers. We consider these mode functions in Fourier space where we scale all momenta by $k_i = \tilde{k}_i \sqrt{m\omega_i^2/2\mu}$ for $i = x, y, z$ with chemical potential μ and trap frequency ω_i in direction i.

Before going into any technical details we state the central result of this paper and discuss how it may be tested experimentally: We show in this paper that the Fourier transform $\tilde{\phi}(\tilde{\mathbf{k}})$ of the mode functions for traps of *arbitrary anisotropy* factorizes into a radial part depending only on the *modulus* of the scaled wave vector

$$\tilde{k} = |\tilde{\mathbf{k}}| = \sqrt{2\mu/m(k_x^2/\omega_x^2 + k_y^2/\omega_y^2 + k_z^2/\omega_z^2)} \tag{0.1}$$

times an angular part $g(\theta_k, \phi_k)$ depending only on the two angles θ_k, ϕ_k of spherical coordinates in scaled Fourier space. Furthermore the radial part does not depend on the trap anisotropy anymore and can be determined explicitly. We simply have

$$\tilde{\phi}(\tilde{\mathbf{k}}) = \frac{j_\nu(\tilde{k})}{\tilde{k}} \times g(\theta_k, \phi_k)\,, \tag{0.2}$$

where j_ν denotes the spherical Bessel-function, the positive integer degree ν depending on the three quantum numbers of the mode functions, which will be specified below. This factorization is remarkable since neither the original problem nor the problem with rescaled momenta are spherically symmetric. In fact it follows from a somewhat hidden conservation law of an operator with spherical symmetry after rescaling as we shall show below.

Before we derive (0.2) we wish to illustrate this property by discussing how it could be observed in an experiment. One possibility to observe density fluctuations is by the direct observation of a condensate by phase-contrast imaging [7,8]. A beam of light off resonace with any atomic transition accumulates a phase shift on its way through the condensate, which is proportional to the density of atoms integrated along its optical path. In phase-contrast imaging the direct beam of light is shifted by $\pi/2$ by passing it through a $\lambda/4$-plate in the focal plane. In the interference pattern between the direct beam and the deflected light on a screen phase differences are thereby turned into amplitude differences. Since typical angles of deflection in the experiment of [8] are only of the order of several mrad we can approximate the paths of deflected light also as straight lines parallel to the optical axis. The observed amplitude differences are therefore proportional to the projection of the three dimensional number density of the condensate on a two dimensional screen. By this technique it is possible to take time resolved sequences of pictures of an oscillating condensate [7], after its excitation via a modulation of the trap with a certain frequency.

By Fourier transformimg these sequences of pictures in time with the known mode-frequency one gets the projection of the density fluctuation mode $\phi(\mathbf{r})$ onto the screen, i.e. the mode function integrated along the direction of the optical axis. The Fourier transform $\tilde{\phi}(\mathbf{k})$ of the mode function for any wave vector $\mathbf{k}$ perpendicular to the optical axis can then be obtained by taking the spatial Fourier transformation of this picture with the wave vector $\mathbf{k}$ in the image-plane. To test relation (0.2) the Fourier transform must be taken for a range of k-values with fixed direction of $\mathbf{k}$, and thus also of the scaled $\tilde{\mathbf{k}}$, calculating the modulus $\tilde{k}$ from (0.1). The chemical potential neccessary for the application of (0.1) must be determined from the number of atoms in the trap. For fixed direction of $\tilde{\mathbf{k}}$ $g(\theta_k, \phi_k)$ only occurs as a constant factor, which of course depends on the direction chosen and which may vanish in some directions.

The scaling introduced for the momenta corresponds to scaling all coordinates with the Thomas-Fermi radius in this direction according to $r_i = \tilde{r}_i \sqrt{2\mu/m\omega_i^2}$ for $i = x, y, z$. By this spatial rescaling an anisotropic harmonic trap is transformed to an isotropic one and pictures of elliptical condensates are transformed to spherical ones. Therefore, instead of calculating the spatial Fourier transform of pictures of mode functions in scaled momenta one can also first unfold these pictures to spherical symmetry and then apply normal Fourier transformation. In an experiment the optical axis probably would be oriented along one axis of the trap, e.g. the y-axis. The simplest case then certainly is to take the orientation of $\tilde{\mathbf{k}}$ either along the z-axis, with $\tilde{k} = k_z\sqrt{2\mu/m\omega_z^2}$, or along the x-axis with $\tilde{k} = k_x\sqrt{2\mu/m\omega_x^2}$. For an arbitrary, but fixed orientation of $\tilde{\mathbf{k}}$ we have to keep the ratio k_x/k_z constant with the modulus given by (0.1). In summary, once having taken a sequence of phase-contrast images for a particular mode, relation (0.2) could be tested easily by a simple analysis.

Now let us derive the property (0.2). For a Bose gas below the Bose-Einstein transition the usual separation of the particle field operator into a classical part $\phi_0(\mathbf{r})$, the condensate wave function, and a residual operator $\hat{\phi}(\mathbf{r})$, describing excitations out of the condensate, yields for the number-density operator

$$\hat{\rho}(\mathbf{r}) = |\phi_0(\mathbf{r})|^2 + \left(\phi_0^*(\mathbf{r})\hat{\phi}(\mathbf{r}) + \phi_0(\mathbf{r})\hat{\phi}^\dagger(\mathbf{r})\right) + \hat{\phi}^\dagger(\mathbf{r})\hat{\phi}(\mathbf{r})\,.$$

Thermal averaging of the last term gives the condensate depletion at $T = 0$ plus the density of thermally excited atoms. At low temperatures for a weakly interacting system the condensate density $|\phi_0(\mathbf{r})|^2$ contains most of the particles, and the second term dominates by far the last term. Restricting ourselves to the low-lying density fluctuations with $\hbar\omega \ll \mu$ [9], one single mode of these dominant density fluctuations with mode frequency ω takes the form [10,11]

$$\phi_0^*(\mathbf{r})\hat{\phi}(\mathbf{r}) + \phi_0(\mathbf{r})\hat{\phi}^\dagger(\mathbf{r}) = i\sqrt{\frac{\hbar\omega}{2g}}(\phi(\mathbf{r})\hat{\alpha}e^{-i\omega t} - \phi^*(\mathbf{r})\hat{\alpha}^\dagger e^{i\omega t})\,.$$

Here g is the atom-atom interaction constant $4\pi\hbar^2 a/m$, with the s-wave scattering length a, and $\phi(\mathbf{r})$ a normalized density fluctuation mode. These density fluctuations and the corresponding velocity fluctuations can be described by hydrodynamic equations of the Euler type. From this description one can derive a wave equation [9,10]

$$-\omega^2\,\phi(\mathbf{r}) = \nabla\cdot\frac{g}{m}|\phi_0(\mathbf{r})|^2\nabla\phi(\mathbf{r}) \tag{0.3}$$

which both the normalized density modes and the velocity fluctuation modes have to fulfill [9,10].

Features of Bose condensates are most readily observable in the limit of large condensates. In this limit the Thomas-Fermi approximation for the condensate wave function $g|\phi_0(\mathbf{r})|^2 = (\mu - U(\mathbf{r}))\Theta(\mu - U(\mathbf{r}))$ is justified [12], which restricts solutions of (0.3) to the interior of the Thomas-Fermi surface $\mu = U(\mathbf{r})$. For condensates in isotropic harmonic traps the spectrum and the eigenfunctions of (0.3) were calculated by Stringari [9]. They depend on a radial and an angular quantum number n_r and l, the spectrum being degenerate with respect to the azimuthal quantum number m_l.

The atom-traps used in the different realisations of Bose condensates in alkali atoms can be decribed in very good approximation by an anisotropic harmonic potential $U(\mathbf{r}) = m(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)/2$. Here we do not restrict ourselves to any axial or spherical symmetry. Due to the anisotropy of the potential new features show up: the azimuthal quantum number m_l is no longer a good quantum number, at energies of the order of μ the system ceases to be strictly integrable, and a large chaotic component can be seen in the classical phase space [13]. However in the limit of low energies ($\hbar\omega \ll \mu$) the systems reduces to an integrable one [16] and an additional constant of motion can be identified [14]

$$\hat{B} = -\sum_i \frac{2\mu}{m\omega_i^2}(\frac{\partial}{\partial r_i})^2 + (\mathbf{r}\cdot\nabla)^2 + 3\mathbf{r}\cdot\nabla\,. \tag{0.4}$$

We shall now show how (0.2) follows from this conserved quantity.

Let us introduce a scaled position-vector $\tilde{\mathbf{r}}$ with dimensionless components $\tilde{r}_i$ all in the range $[0,1]$ by the scaling $r_i = \tilde{r}_i \sqrt{2\mu/m\omega_i^2}$ for $i = x, y, z$. Any harmonic trap potential now becomes $U(\tilde{\mathbf{r}}) = \mu\tilde{r}^2$ and the Thomas-Fermi surface is mapped to the unit sphere. The wave equation (0.3) takes the form

$$-\omega^2\,\phi(\tilde{\mathbf{r}}) = \frac{1}{2}\sum_i \Big[\omega_i^2 \frac{\partial}{\partial\tilde{r}_i}(1-\tilde{r}^2)\frac{\partial}{\partial\tilde{r}_i}\Big]\,\phi(\tilde{\mathbf{r}})\,.$$

Now let us change to scaled Fourier space. Since terms from partial integration don't occur due to the vanishing of the condensate wave function at the condensate boundary we simply can replace $\tilde{r}_j \to i\partial/\partial\tilde{k}_j$ and $\partial/\partial\tilde{r}_j \to ik_j$. The modulus of the scaled momentum $\tilde{k}$ is given by (0.1). After simple rearrangements we get from (0.3)

$$-\omega^2\,\tilde{\phi}(\tilde{\mathbf{k}}) = \sum_i \Big[\frac{1}{2}\omega_i^2\tilde{k}_i^2(1+\nabla_{\tilde{\mathbf{k}}}{}^2) + \omega_i^2\tilde{k}_i\frac{\partial}{\partial\tilde{k}_i}\Big]\,\tilde{\phi}(\tilde{\mathbf{k}})\,.$$

Next we introduce spherical coordinates in scaled Fourier space in the usual way as $\tilde{k}_x = \tilde{k}\sin\theta_k\cos\phi_k$, $\tilde{k}_y = \tilde{k}\sin\theta_k\sin\phi_k$ and $\tilde{k}_z = \tilde{k}\cos\theta_k$. The explicit form of the wave equation then becomes

$$\begin{aligned}
-\omega^2\,\tilde{\phi}(\tilde{k},\theta_k,\phi_k) = \frac{1}{2}\Big(&\big(\omega_x^2\sin^2\theta_k\cos^2\phi_k + \omega_y^2\sin^2\theta_k\sin^2\phi_k + \omega_z^2\cos^2\phi_k\big)\\
&\times\Big[\tilde{k}^2 + \tilde{k}^2(\frac{\partial}{\partial\tilde{k}})^2 + 4\tilde{k}\frac{\partial}{\partial\tilde{k}} + (\frac{\partial}{\partial\theta_k})^2 + \cot\theta_k\frac{\partial}{\partial\theta_k} + \frac{1}{\sin^2\theta_k}(\frac{\partial}{\partial\phi_k})^2\Big]\\
&+ \sin 2\theta_k\big(\omega_x^2\cos^2\phi_k + \omega_y^2\sin^2\phi_k - \omega_z^2\big)\frac{\partial}{\partial\theta_k} - \sin 2\phi_k\big(\omega_x^2-\omega_y^2\big)\frac{\partial}{\partial\phi_k}\Big)\,\tilde{\phi}(\tilde{k},\theta_k,\phi_k)\,.
\end{aligned}$$

We see that the radial part, depending on $\tilde{k}$ only, can be separated from this equation which justifies the factorized form of the separation ansatz (0.2). Our point now is that the separated equation for the radial part simply is the eigenvalue equation for the operator $\hat{B}$ in the scaled Fourier representation. To see this let us transform $\hat{B}$ to the $\tilde{\mathbf{k}}$-representation. Starting in coordiante space again, scaling the coordinates in (0.4) first, and then changing to Fourier representation we get

$$\begin{aligned}
\hat{B} &= -\tilde{\nabla}^2 + (\tilde{\mathbf{r}}\cdot\tilde{\nabla})^2 + 3\tilde{\mathbf{r}}\cdot\tilde{\nabla}\\
&= \tilde{k}^2 + \tilde{k}^2(\frac{\partial}{\partial\tilde{k}})^2 + 4\tilde{k}\frac{\partial}{\partial\tilde{k}}\,,
\end{aligned}\qquad (0.5)$$

which is indeed precisely the operator occuring in the radial part of the wave equation. Remarkably the eigenvalue-equation of $\hat{B}$ does not depend on the frequencies ω_i anymore, its spectrum and its solutions are the same for arbitrary anisotropies.

The eigenfunctions of $\hat{B}$ in the $\tilde{\mathbf{k}}$-representation can be found in terms of spherical Bessel-functions j_ν

$$\hat{B}\;\frac{j_\nu(\tilde{k})}{\tilde{k}} = (\nu-1)(\nu+2)\,\frac{j_\nu(\tilde{k})}{\tilde{k}}\,.$$

The spectrum of $\hat{B}$ was previously derived in [14], however, without stating the eigenfunctions. In the notation of [14] for axially symmetric traps one has $\nu - 1 = n + m_l$, where m_l

denotes the azimuthal quantum number. Previous solutions of (0.3) started with a polynomial ansatz in Cartesian coordinates [9,15,16] for $\phi(\mathbf{r})$. We note that $\nu - 1$ is simply the highest degree of such a polynomial solution. For the general anisotropic case the dependence of ν on the three quantum numbers is given in [16]. In terms of the quantum numbers of the isotropic problem [9] one has $\nu - 1 = 2n_r + l$. For this isotropic case (0.2) was already derived by Fetter [17] by direct Fourier transformation of the spatial mode functions. Since the conserved quantity $\hat{B}$ does not depend on the trap frequencies $\omega_x, \omega_y, \omega_z$, subspaces of the Hilbert space for different values of the quantum number ν remain orthogonal for all anisotropies. So the same dependence on the modulus of the momentum $\tilde{k}$ as for the isotropic case holds for all anisotropies.

Let us remark now on the range of validity of our discussion which is limited by our use of the approximate low-frequency form (0.3) of the wave equation and the Thomas-Fermi approximation. For the description by the wave equation (0.3) we have to require $\hbar\omega \ll \mu$. A second condition arises from the finite width of the boundary layer $l_i = (2\mu/m\omega_i^2)^{1/2}(\hbar\omega_i/4\mu)^{2/3}$ for the directions $i = x, y, z$, limiting the momenta $\mathbf{k}$ for which the Thomas-Fermi approximation can still be used: There is a maximum momentum $k_{li} = 2\pi/l_i$, or in scaled variables $\tilde{k}_{li} = 2\pi(4\mu/\hbar\omega_i)^{2/3}$, for which (0.2) is valid. For the cigar shaped condensate of [7] this condition is in radial direction ($\tilde{k}_{lr} \approx 140$) more limiting than in axial direction ($\tilde{k}_{la} \approx 770$). Both limiting values place no restrictions on the experimental observation for the low lying levels, however for higher lying levels they may be relevant. Let us consider the case of axial symmetry somewhat more generally: For a level with quantum numbers (n, j, m_l) in the notation of [14] relation (0.2) can be observed best in the range of the first maximum $k_{\max}$ of the Bessel function $j_{n+m+1}(k)$, scaling as $k_{\max} \sim (n + m_l)$ for large n, m_l. If we have $n \gg m_l$, then $\omega \sim n$ and both limiting conditions on the energy $\omega_{njm} \ll \mu$ and on the momentum $k_{\max} \ll k_l$ roughly coincide. However if $n \ll m_l$, we have $\omega \sim m_l^{1/2}$ and the condition $k_{max} \gg k_l$ is the more limiting one.

As an alternative experiment to determine the collective excitations and to observe (0.2) one might also think of inelastic scattering of light [10,17]. Unfortunately, this turns out to be hardly feasible, as we now show. Light scattered from an atomic gas off resonance with any atomic transition couples to the number density of atoms. The cross section of elastic scattering of light is proportional to the square of the spatial Fourier-transform of the equilibrium density $|\phi_0(\mathbf{r})|^2$. For an inhomogenous system elastic scattering occurs in a finite angle, fixed by the condensate size. Density fluctuations can be seen as inelastic scattering of light. The cross section of light shifted in frequency by ω via scattering from the density fluctuation ϕ is again proportional to the Fourier-transform of this mode function [10]. One might hope to use this to test relation (0.2). For scattering transfering the wave vector $\mathbf{k}$ and at $T = 0$ we get as a ratio of elastic to inelastic cross section $(N\,\mu/\hbar\omega) \times |\tilde{\rho}_0(\mathbf{k})/\tilde{\phi}(\mathbf{k})|^2$, which unfortunately turns out as particularly unfavorable both in the limit $\hbar\omega \ll \mu$ and in the limit of large particle numbers.

The larger the transferred momentum $\mathbf{k}$, the better inelastic scattering can be distinguished from the background of elastic scattering. The maximum momentum, for which (0.2) is valid, is the momentum $\tilde{k}_l$ determined by the boundary width. For present experiments [4,7] this is by a factor $2 - 5$ smaller than the largest momentum transferable by light scattering, occuring in the case of back-scattering. Therefore the inelastic cross section from (0.2) describes scattering for not too large angles of deflection only. For the maximum mo-

mentum permissible by our approximations the elastic cross section is of the order of $1/\tilde{k}_l^6$, which follows from the (on this scale) discontinuous first derivative of $|\phi_0|^2$ at the boundary. For small quantum numbers the inelastic cross section from (0.2) scales as $1/\tilde{k}_l^4$. So collecting the prefactors we get as a minimum overall ratio from elastic to inelastic scattering $N\,(\mu/\hbar\omega)(\hbar\omega_i/4\mu)^{2/3}$ for inelastic scattering from level ω in direction i. For the experiments of [7] this ratio exceeds 10^4 for all directions for the low lying levels. This unfavorable ratio probably precludes the possibility to test relation (0.2) by inelastic light scattering.

In conclusion we have shown that the scaled Fourier-transform of the hydrodynamic mode functions factorizes in a radial part and an angular part, even for harmonic traps of arbitrary anisotropy. The radial part is fixed by only one quantum number and has the simple explicit form (0.2). This result was derived from a previously found conservation law. The eigenvalues and eigenfunctions of the conserved operator $\hat{B}$ determine completely the radial mode function in scaled Fourier space . We have discussed the range of validity of this property and have suggested how it could be observed in experiments using phase-contrast imaging.

M.F. thanks H. Schomerus for a valuable discussion. This work has been supported by the Deutsche Forschungsgemeinschaft through the Sonderforschungsbereich 237 "Unordnung und große Fluktuationen".

REFERENCES

[1] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wiemann, and E. A. Cornell, *Science* **269**, 198 (1995).

[2] C.C. Bradley, C.A. Sackett, J.J. Tollett, and R.G. Hulet, *Phys. Rev. Lett.* **75**, 1687 (1995).

[3] K.B. Davis, M.O. Mewes, M.R. Andrews, N.J. van Druten, D.D. Durfee, D.M. Kurn, and W. Ketterle, *Phys. Rev. Lett.* **75**, 3969 (1995).

[4] D. S. Jin, J. R. Ensher,M. R. Matthews, C. E. Wiemann,and E. A. Cornell, *Phys. Rev. Lett.* **77**, 420 (1996).

[5] M.O. Mewes, M. R. Andrews, N. J. van Druten, C. G. Townsend, and W. Ketterle, *Phys. Rev. Lett.* **77**, 988 (1996).

[6] D. S. Jin, M. R. Matthews, J. R. Ensher, C. E. Wiemann, and E. A. Cornell, *Phys. Rev. Lett.* **78**, 764 (1997).

[7] D. M. Stamper-Kurn, H. J. Miesner, S. Inouye, M. R. Andrews, and W. Ketterle, cond-mat/9801262.

[8] M. R. Andrews, M.-O. Mewes,N. J. van Druten,D. S. Durfee, D. M. Kurn,W. Ketterle, *Science* **273** 84 (1996).

[9] S. Stringari, *Phys. Rev. Lett.* **77**, 2360 (1996).

[10] W.-C. Wu, A. Griffin, *Phys. Rev. A.* **54**, 4204 (1996).

[11] A. L. Fetter, *Phys. Rev. A* **53**, 4245 (1996);

[12] G. Baym and C. J. Pethick, *Phys. Rev. Lett.* **76**, 6 (1996)

[13] M. Fliesser, A. Csordás, R. Graham, P. Szépfalusy, *Phys. Rev* **A 56**, 4879 (1997)

[14] M. Fliesser, A. Csordás, R. Graham, P. Szépfalusy, *Phys. Rev.* **A 56**, R2533 (1997)

[15] P. Öhberg,E. L. Surkov,I. Tittonen,S. Stenholm, M. Wilkens and G. V. Shlyapnikov, *Phys. Rev.* **A 56**, R3346 (1997)

[16] A. Csordás, R. Graham, cond-mat/9809002

[17] A. L. Fetter,D. Rokhsar, *Phys. Rev. A* **57**, 1190 (1998);